Inaccurate Mental Addition and Subtraction: Causes and Compensation

Ann M. Heirdsfield
Centre for Mathematics and Science Education, Queensland University of Technology
<a.heirdsfield@qut.edu.au>

This paper reports on a study of seven Year 3 students' diminished performance in addition and subtraction mental computation. Although all students were identified as being inaccurate, three students used some variety of mental strategies, while the other students used only one strategy that reflected the written procedure for each of the addition and subtraction algorithms taught in the classroom. Interviews were used to identify students' knowledge and ability with respect to number sense (including numeration, number and operations, basic facts, estimation), metacognition, affects, and memory. Two conceptual frameworks were developed, one representing the flexible mental computers, and the other representing the inflexible mental computers. These frameworks identified factors and relationships between factors that influence flexibility in these inaccurate mental computers. The frameworks were compared with a framework of an ideal mental computer. These frameworks showed that inaccuracy resulted from disconnected and deficient cognitive, metacognitive, and affective factors; and in some cases might have been affected by deficient short-term memory. It appeared that students' choices of mental strategies resulted from different forms of compensation for varying levels of deficiencies.

Researchers and educators have stressed the importance of including mental computation in number strands of mathematics curricula (e.g., McIntosh, 1996; Reys & Barger, 1994; Sowder, 1990; Treffers & de Moor, 1990; Willis, 1990). Reasons for its inclusion are that mental computation: (1) enables children to learn how numbers work, make decisions about procedures, and create strategies (e.g., Reys, 1985; Sowder, 1990); (2) promotes greater understanding of the structure of number and its properties (Reys, 1984); and (3) can be used as a "vehicle for promoting thinking, conjecturing, and generalizing based on conceptual understanding" (Reys & Barger, 1994, p. 31). In effect, mental computation promotes number sense (National Council of Teachers of Mathematics, 1989; Sowder, 1990). In fact, Willis (1992) suggested that mental computation should be the main form of computation, with written computation to serve as memory support. However in the existing Queensland curriculum document, *Years 1 to 10 mathematics teaching, curriculum and assessment guidelines* (Department of Education, Queensland, 1987), addition and subtraction mental computation is not mentioned.

Proficiency in mental computation has been the focus of several research projects (e.g., Beishuizen, 1993; Heirdsfield, 1996; McIntosh & Dole, 2000; Reys, Reys, Nohda, & Emori, 1995). In The Netherlands, where mental computation is taught before written computation, mathematics programs emphasise the use of aggregation (N10) as a more efficient mental strategy. However, weaker students tended to use less efficient separation strategies (Beishuizen, 1993). Reys, Reys, Nohda, and Emori (1995) found that accuracy in mental computation was associated with strategies other than mental image of pen and paper algorithm. In contrast to these findings, McIntosh and Dole (2000) reported higher accuracy when students employed mental image of pen and paper algorithm than when they employed alternative mental strategies (although these alternative strategies revealed number sense). Heirdsfield (1996) also found that accuracy in mental computation did not need to be accompanied by employment of a variety of efficient mental strategies.

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 334-341). Sydney: MERGA. ©2002 MERGA Inc.

Therefore, while some research appears to indicate that accuracy in mental computation is a result of efficient mental strategies, other research has reported accuracy as a result of employment of strategies that reflect pen and paper algorithms.

Research undertaken by this author investigated mental computers and the factors that supported accuracy and flexibility (Heirdsfield, 1998, 2001a, 2001b). This study investigated the part played by number sense knowledge (e.g., numeration, number facts, estimation and effects of operations on number), metacognition (metacognitive knowledge, strategies and beliefs), affects (e.g., beliefs, attitudes), and memory (working memory – Baddeley, 1986 – and long term memory) in mental computation. It showed that students proficient in mental computation (accurate and flexible) possessed integrated understandings of number facts (speed, accuracy, and efficient number facts strategies), numeration, and number and operation. These proficient students also exhibited some metacognitive strategies and metacognitive beliefs, and affects (e.g., beliefs about self and teaching) that supported their mental computation. Further, proficient mental computers had good short-term recall to hold interim calculations and recall number facts, and well developed central executive (in working memory) to attend to the demanding task of mental computation and retrieve strategies and facts from a well-connected knowledge base in long term memory. Proficient mental computers chose alternative and efficient strategies, as they possessed extensive and connected knowledge bases to support these strategies. Thus, there was evidence of the importance of connected knowledge, including domain specific knowledge, and metacognitive strategies, affects and memory for proficient mental computation. As a result of this study, a conceptual framework identifying associated factors involved in proficient mental computation was developed (Heirdsfield, 2001b).

This leads to the question as to what are the effects on mental computation of less knowledge and fewer connections? It would be expected that one effect would be less accuracy. The purpose of this paper is to report on seven students who were inaccurate in addition and subtraction mental computation, but they exhibited differing levels of flexibility of mental strategies. These students were participants in a study that investigated addition and subtraction mental computation in Year 3 students (Heirdsfield, 2001c). Conceptual frameworks for these inaccurate students are developed and compared with a framework for the "ideal" mental computer (flexible and accurate) that was developed in the large study from which these students were drawn.

Method

Participants

The participants were seven students, selected from a population of sixty Year 3 children from three classrooms, representing two Independent schools in Brisbane. Both schools served students from high and middle socio-economic areas. The students were selected on the basis of an interview that identified accuracy and flexibility in mental computation. Although all students were considered inaccurate, three students employed some variety of strategies; therefore they were identified as flexible. The other four students employed a single strategy consistently. Therefore, these four students were identified as inflexible.

Instruments

The students were presented with a series of tests and indepth interviews. These were number fact knowledge, mental computation (one-, two-, and three-digit addition and subtraction), computational estimation, numeration, effects of operation on number, and memory. In order to choose neuropsychological tests relevant to these aspects, Lezak (1995) was consulted. The neuropsychological tests, which were used with the Year 3 students, were aimed at investigating short-term recall and executive functioning. The tests were modifications of a Digit Span Test (short-term recall) and a maze test (central executive, e.g., planning and attention).

Further, questions were asked addressing attribution, self-efficacy, beliefs, and metacognition. The students were also required to complete the *Student Preference Survey* (*SPS*) (McIntosh, 1996), to identify whether they would and could solve computational tasks mentally.

Procedure

The students were withdrawn from their classroom on a one to one basis, and interviewed in a quiet room. The interviews were videotaped, and each interview session lasted for no more than 30 minutes at a time. Because of the variety of aspects covered, each child received four interview sessions (three sessions in the pilot study).

Analysis

For the purposes of identifying flexibility in mental computation, mental computation strategies were identified using the categorisation scheme (based on Beishuizen, 1993; Cooper, Heirdsfield, & Irons, 1996; Reys, Reys, Nohda, & Emori, 1995) that divided the strategies into the following categories: (1) *separated* (e.g., 38+17: 30+10=40, 8+7 = 15 = 10+5, 40+10+5 = 55 – *separation left to right*; 38+17: 30+10=40, 40+8=48, 48+9=57 – *cumulative sum*); (2) *aggregation* (e.g., 38+17: 38+10=48, 48+7 = 55); (3) *wholistic* (e.g., 38+17 = 40+17-2 = 57-2 = 55); and (4) *mental image of pen and paper algorithm* – following an image of the formal setting out of the written algorithm (taught to almost automaticity in the schools the students attended).

The analysis of the interviews incorporated three stages. First, each interview for each student was analysed separately. Second, relationships across interviews for each student were considered (e.g., whether understanding of the effect of operations on number was used for mental computation, whether the same number facts strategies were employed in both the number facts test and in the mental computation interview). Third, analysis compared commonalities and differences across students.

Mental computation responses were analysed for strategy choice, flexibility, accuracy, understanding of number facts, computational estimation, numeration, and the effects of operations on number. It was also noted whether the students could access alternative mental strategies, when encouraged to do so. Number facts were analysed for accuracy, speed and strategy choice. Estimation strategies were identified and proficiency and flexibility were noted. Analysis of students' responses to numeration tasks was based on Ross's five levels (1986), which included canonical and noncanonical understanding of number. Also, evidence of multiplicative understanding (e.g., ten tens are the same as one hundred) was investigated. The tasks addressing the effect of operations on number were analysed for understanding of arithmetic properties (e.g., associativity, inverse, the effect of changing the addend and subtrahend) as they apply to computational relationships (e.g., 70-43=27, ∴70-44=26).

For the analysis of the memory tests, Lezak (1995) was consulted. As so few students were interviewed, it was decided to compare individuals' raw scores for the Digit Span Test, and note any trends with memory problems evident in mental computation tasks. The maze completion times and the number of errors, such as retracing lines or entering blind alleys, were recorded. Although there might be a tenuous link between executive functioning in completing mazes and executive functioning in completing mental computation tasks, the fact that a student could attend to a task and plan would affect mental functioning in any domain. Evidence for planning and decision-making was compared with the same in the mental computation interviews.

Results

Comparison of Flexible and Inflexible Inaccurate Mental Computers

Although all students were inaccurate, there were differences in the mental computation strategies they employed. These strategies appeared to be used to compensate for limited and disconnected knowledge. A comparison between the flexible and inflexible students is made in Figure 1.

	Not accurate and flexible (n=3)	Not accurate and inflexible (n=4)
Mental computation	Strategies: Separation strategies only (left to right, right to left, cumulative). Alternative strategies: yes (including wholistic), but not always successful.	Strategies: Used mental image of pen and paper algorithm. Alternative strategies: yes, but rarely successful
Number facts	Accuracy: accurate for addition, generally slow and inaccurate for subtraction Strategies: Derived Facts Strategies (DFS), recall, and count. Used count mostly in mental calculations.	Accuracy: inaccurate and slow Strategies: DFS (very few though), recall, and count. Used count mostly in mental calculations.
Computational estimation	Varied	Mostly poor.
Numeration	Varied, canonical, noncanonical, proximity of numbers.	Generally poor, mostly only canonical understanding with material.
Number and operation	Mostly poor.	Mostly poor.
Metacognition	Some strategies, mostly inaccurate beliefs.	Mostly no strategies, inaccurate beliefs.
Affects	Varied beliefs and no strong beliefs evident.	Varied beliefs.
Working Memory:	Evidence of central executive (planning and attention). ST recall score: ≥ 6	Evidence of diminished central executive. ST recall score: ≤ 5

Figure 1. Comparison of flexible and inflexible inaccurate mental computers.

In general, both groups of students lacked sufficient understanding of number facts, estimation, numeration, and number and operation to support advanced mental computation strategies. Although the students were unsuccessful with the taught procedures, the flexible students attempted to compensate by inventing strategies, although most (but not all) of these strategies were not high order strategies. Some number facts strategies, numeration understanding, and metacognitive strategies assisted the employment of alternative mental calculation strategies. Thus, the flexible students attempted to compensate for their lack of procedural understanding, but their knowledge was diminished and disconnected. On the other hand, the inflexible students attempted to compensate by employing the teacher taught strategy, which required little conceptual understanding (and also provided a mental image to support memory). However, because of a lack of procedural understanding, errors still resulted. None of the inflexible students held accurate perceptions of their mental computation abilities. Also, poor short-term recall and diminished executive functioning compounded these deficiencies. In other words, their knowledge was so deficient and disconnected, even strategies that had been taught (but not learnt) could not be followed.

Results of Flexible Students

The three flexible students used some variety of strategies, but few high order strategies. Most strategies were *separation* (*left to right*, *right to left*, and *cumulative sum/difference*). Although, one student also attempted to employ *aggregation left to right* and *wholistic* strategies for addition in the indepth interview.

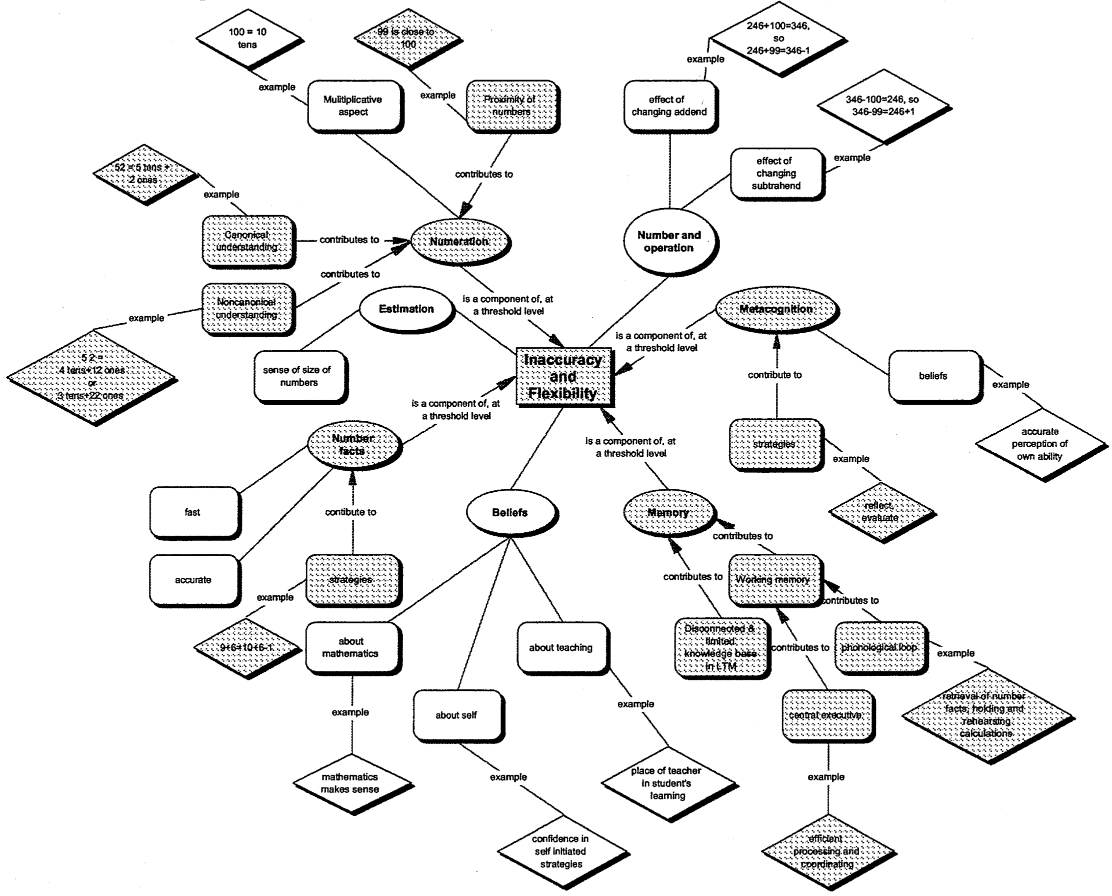

Figure 2. Conceptual framework for inaccurate and flexible computation.

In general, the students lacked sufficient understanding of number facts (although some efficient strategies were employed), numeration, and number and operation to support advanced mental computation strategies. Estimation was poor and, thus, did not support mental computation. Further, these students were unsuccessful with the taught computational procedures, so they compensated by inventing strategies, although most (but not all) of these strategies were not high order strategies. Some numeration understanding and metacognitive strategies assisted mental calculation using alternative strategies. The students were unable to use teacher taught procedures to compensate for their deficiencies.

Thus, lack of procedural understanding of the pen and paper procedures resulted in the students' inventing mental strategies. However, as they did not possess sufficient understanding of number facts and number and operation, they were rarely successful. Some numeration understanding and metacognitive strategies assisted the invention of mental strategies, however, there was insufficient understanding to support high-level mental strategies. Thus, they attempted to *compensate* for their lack of procedural understanding, but their knowledge was *disconnected.* A conceptual framework for the inaccurate and flexible students is presented in Figure 2. This framework was developed in comparison to that of a proficient mental computer (Heirdsfield, 2001b). To show missing factors, cells are left unshaded. Dappled effect in cells indicates that knowledge was at a threshold level (compared with proficient mental computers). Arrowheads are eliminated to show disconnected knowledge.

Results of Inaccurate and Inflexible Students

The four inflexible students predominantly employed *mental image of pen and paper algorithm*. They all reported "seeing" or imagining the vertical format of the pen and paper algorithms.

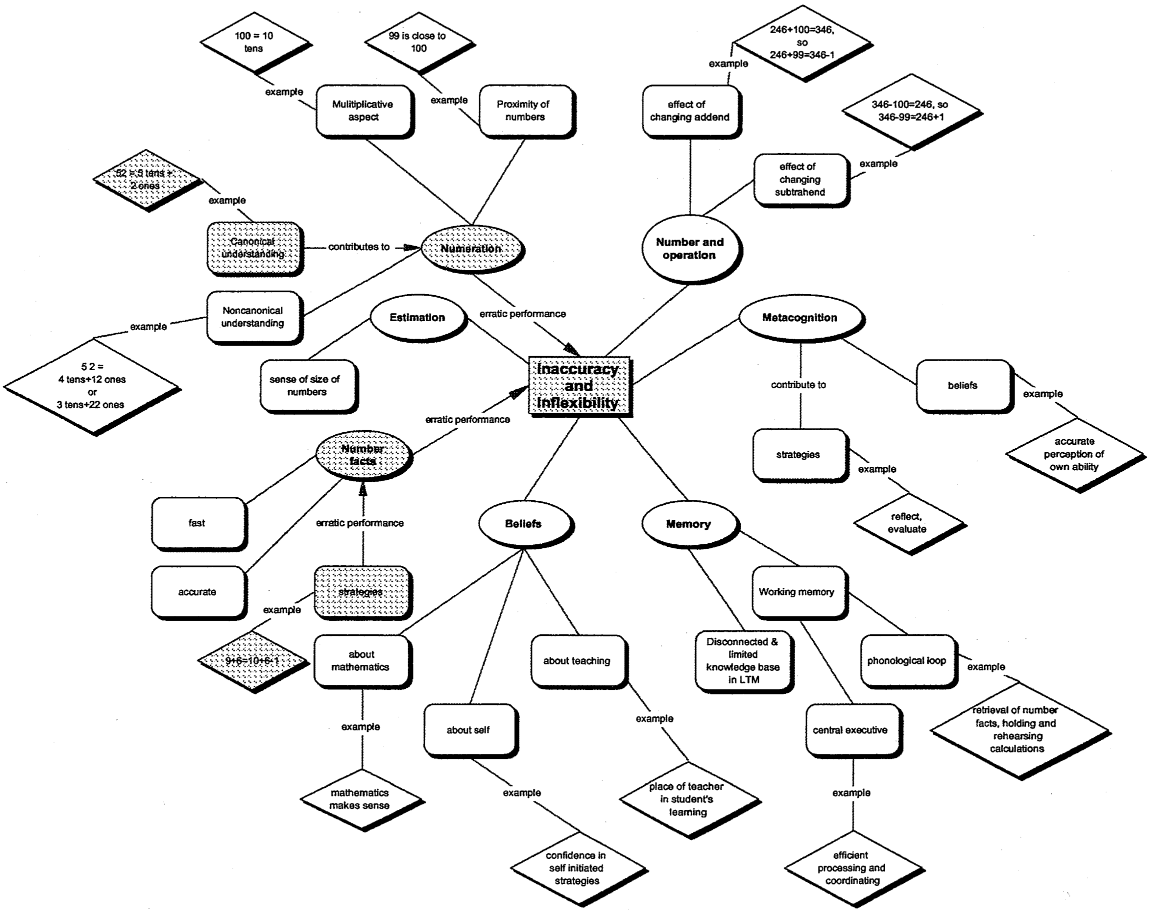

Figure 3. Conceptual framework for inaccurate and inflexible mental computation.

In general, poor number facts knowledge (i.e., inaccurate), and poor understanding of estimation, numeration, and number and operation contributed to inaccuracies in mental computation, and the inability to access alternative strategies. Also, poor short-term recall and diminished executive functioning compounded these deficiencies. To compensate for a poor knowledge base and memory overload, they attempted to employ an automatic strategy, which required little conceptual understanding (and also provided a mental image to support memory). However, because of a lack of procedural understanding, errors still resulted. None of these inflexible students held accurate perceptions of their mental computation abilities. Finally, the inflexible students exhibited little or no understanding of any of the factors investigated in relation to mental computation, resulting in *deficient and disconnected* knowledge.

In summary, the students exhibited *deficient and disconnected* knowledge. To compensate, the students resorted to an automatic strategy, but lack of procedural understanding and other deficiencies resulted in inaccurate application of this strategy. A conceptual framework for the inaccurate and inflexible students is presented in Figure 3. This framework was developed from that of the proficient mental computer. To show missing factors, cells are left unshaded. Dappled effect in cells indicates that knowledge was at a threshold level (compared with proficient mental computers). Arrowheads are eliminated to show disconnected knowledge.

Discussion

Some key differences between the two groups of inaccurate computers were number fact strategies, numeration understanding, metacognitive strategies, and working memory. It was interesting how different forms of compensation resulted from different knowledge bases.

It would seem obvious that knowing number facts by immediate recall would aid in mental computation, as there would be less memory load. Many of the flexible students (and to less extent the inflexible students) used efficient *Derived Facts Strategies* (c.f., *count*) when they could not recall number facts in the number facts test. However, they resorted to *count* for calculating interim calculations during mental computation. The reason might lie in the extra load placed on working memory when *DFS* are employed. *Count* seems to be a more primitive strategy that they and the inflexible students resorted to for interim calculations. Therefore, permitting students to use pen and paper would alleviate working memory load due to lack of number fact knowledge. Further, students should be encouraged to develop efficient *DFS*, where understanding rather than speed is a focus.

Some numeration understanding supported alternative low-level mental computation strategies. However, overall, numeration understanding was lacking in both groups of students. Whether teaching numeration as prerequisite knowledge is necessary or whether numeration understanding can be improved in conjunction with computational understanding is open to question. The success of teaching experiments (e.g., Buzeika, 1999; Kamii, 1989) where students are encouraged to formulate and discuss self-developed computational strategies seems to indicate that the development of efficient computational strategies has a positive effect on the development of numeration.

Most importantly, students need to realise that mathematics should make sense. More emphasis should be placed on students making meaning, rather than students learning procedures that are not/cannot be successfully employed.

References

Baddeley, A. D. (1986). *Working memory.* Oxford: Oxford University Press.

Beishuizen, M. (1993). Mental strategies and materials or models for addition and subtraction up to 100 in Dutch second grades. *Journal for Research in Mathematics Education, 24*(4), 294-323.

Buzeika, A. (1999). Invented algorithms: Teachers face the challenge. In J. M. Truran & K. M. Truran (Eds.), *Making the difference. Proceedings of the Sixteenth Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 128-134). Sydney: MERGA.

Cooper, T. J., Heirdsfield, A. M., & Irons, C. J. (1996). Children's mental strategies for addition and subtraction word problems. In J. Mulligan & M. Mitchelmore (Eds.), *Children's number learning* (pp. 147-162). Adelaide: Australian Association of Mathematics Teachers, Inc.

Department of Education, Queensland. (1987). *Years 1 to 10 mathematics teaching, curriculum and assessment guidelines*. Brisbane: Government Printer.

Heirdsfield, A. M. (1996). *Mental computation, computational estimation, and number fact knowledge for addition and subtraction in year 4 children*. Unpublished master's thesis, Queensland University of Technology, Brisbane.

Heirdsfield, A. M. (1998, July). Flexible/inflexible: Clare and Mandy's story. In C. Kanes, M. Goos, & E. Warren. (Eds.), *Teaching mathematics in new times.* (Proceedings of the 21st annual conference of the Mathematics Education Research Group of Australasia, pp. 241-249). Gold Coast, Australia: MERGA.

Heirdsfield, A. M. (2001a, July). Integration, compensation and memory in mental addition and subtraction. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education,* (Vol 3, pp. 129-136). The Netherlands: Drukkerij Wilco.

Heirdsfield, A. M. (2001b, July). Integration and compensation in accurate mental computation. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.), *Numeracy and beyond* (Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia, Vol 2, pp. 273-280). Sydney: MERGA.

Heirdsfield, A. M. (2001c). *Mental computation: The identification of associated cognitive, metacognitive and affective factors*. Unpublished doctoral thesis, Queensland University of Technology, Brisbane.

Kamii, C. (1989). *Young children continue to reinvent arithmetic - 2nd grade: Implications of Piaget's theory*. New York: Teachers College Press.

Lezak, M. D. (1995). *Neuropsychological assessment.* Third edition. New York: Oxford University Press.

McIntosh, A. (1996). Mental computation and number sense of Western Australian students. In J. Mulligan & M. Mitchelmore (Eds.), *Children's number learning* (pp. 259-276). Adelaide: Australian Association of Mathematics Teachers, Inc.

McIntosh, A., & Dole, S. (2000). Mental computation, number sense and general mathematics ability: Are they linked? In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000.* (Proceedings of the twenty-third annual conference of the Mathematics Education Research Group of Australasia Incorporated, pp. 401-408). Perth: MERGA.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM.

Reys, B. J. (1985). Mental computation. *Arithmetic Teacher, 32*(6), 43-46.

Reys, B. J., & Barger, R. H. (1994). Mental computation: Issues from the United States perspective. In R. E. Reys & N. Nohda (Eds.), *Computational alternatives for the twenty-first century.* Reston, Virginia: The National Council of Teachers of Mathematics.

Reys, R. E. (1984). Mental computation and estimation: Past, present and future. *Elementary School Journal, 84*(5), 546-557.

Reys, R. E., Reys, B. J., Nohda, N., & Emori, H. (1995). Mental computation performance and strategy use of Japanese students in grades 2, 4, 6, and 8. *Journal for Research in Mathematics Education, 26*(4), 304-326.

Ross, S. H. (1986). *The development of children's place value numeration concepts in grades two through five.* Paper presented at the annual meeting of the American Educational Research Association. San Francisco, April.

Sowder, J. T. (1990). Mental computation and number sense. *Arithmetic Teacher, 37*(7), 18-20.

Treffers, A., & de Moor, E. (1990). *Towards a national mathematics curriculum for the elementary school. Part 2. Basic skills and written computation.* Zwijssen, Tilburg, The Netherlands.

Willis, S. (1990). Numeracy and society: The shifting ground. In S. Willis (Ed.), *Being numerate: What counts?* (pp. 1-23). Melbourne: ACER.

Willis, S. (1992). The national statement on mathematics for Australian schools: A perspective on computation. In C. J. Irons (Ed.), *Challenging children to think when they compute* (pp. 1-13). Brisbane: Centre for Mathematics and Science Education, Queensland University of Technology.